

The Radon transform on half sphere

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Abstract. The Radon transform that integrates a function in \mathcal{S}^n , the n -dimensional sphere, over totally geodesic submanifolds with codimension 1, the great circles, and the dual Radon transform are investigated in this paper. Inversion formulas, range spaces and null spaces are given.

0. Introduction

In this article we investigate the Radon transform and its dual on the sphere. The Radon transform on the 2-sphere was first considered by Funk in [6] and then generalized by Helgason [11]. The main question is the invertibility of the transform and due to [7], [9], [12] some inverse formulas are known in odd and even dimension.

The method of the spherical harmonics was used several times in the case of Euclidean space [16], [18], [19] and proved very useful to answer almost any question. We shall show out in this paper that this theory works just as good on the sphere as it did on the Euclidean and hyperbolic space [15], [16]. Our considerations lead us to new inversion formulas and to a better understanding of the Radon transform and its dual. Furthermore, our closed inversion formulas settle the question [13] whether the inverse transform is local or global by giving its precise structure. (For newer results this path lead to, see references quoted with letters.)

If f is a function defined on Σ^n , the open half of the n -dimensional sphere \mathcal{S}^n , the Radon transform of f is a function Rf defined on the half great circles, the totally geodesic submanifolds of Σ^n with codimension 1. The value of Rf at a given half great circles is the integral of f over that half great circles.

The dual Radon transform R^*F of a function F defined on the set of half great circles is a function on Σ^n . The value of R^*F at a given point X is the integral of F over the set of half great circles passing through X by the surface measure of

AMS Subject Classification (2000): 44A05.

the unit sphere of the tangent space at X (the normals of the half great circles at X project the surface measure of this unit sphere to a measure on the set of the great circles through X).

We modify a little bit the original definition of the dual transform [13] that we call as boomerang transform. The points of the half great circles nearest to the origin define an $n - 1$ dimensional submanifold and by definition the boomerang transform integrates a function f defined on Σ^n just over this submanifold by the projected measure of the unit sphere in $T_X \Sigma^n$. This submanifold could also be defined as the set of the points from which the geodesic segment joining the origin and X can be seen at 90° degrees.

After producing explicit formulas for the Radon and the boomerang transform in Section 1. we invert them in Section 2. Our inversion formulas are obtained in terms of an expansion in spherical harmonics and give also the so called support theorems as a byproduct. This support theorem is also proved in [17] and for $n = 2$ in a stronger form in [14].

Our aim in Section 3. is to prove continuity results and to characterize the ranges and the null spaces of the Radon and boomerang transformations on certain classes of square integrable functions. Similar results can be found in [20].

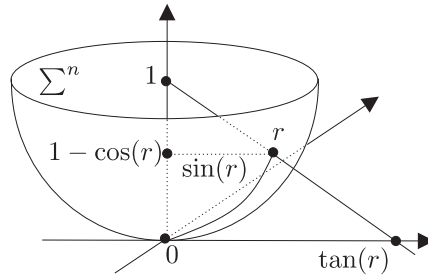
Finally in Section 4. we give two new closed inversion formulas valid in odd and even dimension respectively. Our odd dimensional formula exhibits some similarities to Helgason's formula in [13] and the even dimensional one involves a distribution of the Chauchy principal value type in analogy to the Euclidean space. This proves that the inverse transform is local in odd dimensions and global in even dimensions. This was proved before only implicitly in [17].

Although we shall state our results for the half sphere, they are valid in fact for the elliptic space, because of the trivial symmetry of the original Radon transform.

1. Calculation of the transforms on Σ^n

For the calculation we use the bottom half of the unit sphere in \mathbb{R}^{n+1} with center $(0, 0, \dots, 0, 1)$ as Σ^n .

Let $\mu: \mathbb{R}^n \rightarrow \Sigma^n$ be the parameterization of Σ^n with \mathbb{R}^n given by the projection from the center $(0, 0, \dots, 0, 1)$. Since \mathbb{R}^n and Σ^n are conformally related at the origin the geodesic polar coordinatization of \mathbb{R}^n and Σ^n differ only in the radial coordinates. With $i_{\mathbb{R}}: S^{n-1} \times [0, \infty) \rightarrow \mathbb{R}^n$ and $i_{\Sigma}: S^{n-1} \times [0, \frac{\pi}{2}) \rightarrow \Sigma^n$ denoting the respective (polar)coordinatization this means that $i_{\Sigma}^{-1} \circ \mu \circ i_{\mathbb{R}}(\omega, r) = (\omega, \bar{\mu}(r))$, where $\bar{\mu}: [0, \infty) \rightarrow [0, \frac{\pi}{2})$ is $\bar{\mu}(r) = \arctan(r)$ and $\bar{\mu}^{-1}(r) = \tan(r)$ as the Figure 1 shows.

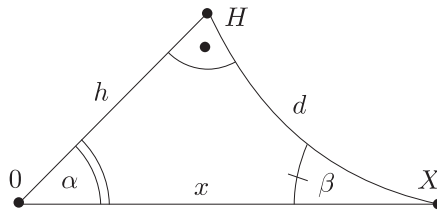
**Figure 1.**

Now let us take a rectangular geodesic triangle on the half sphere Σ^n . One can see on Figure 2 this situation, where $h = d(O, H)$, $d = d(H, X)$, $x = d(O, X)$ (d =distance), α is the angle $HOX\angle$ and β is the angle $HXO\angle$. Then the sine and the cosine rules on sphere imply that

$$(1.1) \quad \tan x = \frac{\tan h}{\cos \alpha}$$

$$(1.2) \quad \sin d = \frac{\sin \alpha}{\sqrt{1 + \cos^2 \alpha \cot^2 h}}$$

$$(1.3) \quad \cos \beta = \frac{\sin \alpha}{\sqrt{1 + \cos^2 \alpha \tan^2 x}}$$

**Figure 2.**

To get explicitly the Radon and boomerang transforms in Σ^n , we parameterize the set of totally geodesic submanifolds of Σ^n by $S^{n-1} \times [0, \frac{\pi}{2})$ in such a way that $\xi(\omega, h)$ denotes the totally geodesic submanifold ξ perpendicular to the geodesic passing through the origin with tangent vector ω at distance h from the origin.

Lemma 1.1. *Let $f \in L^2(\Sigma^n)$. Then its Radon transform is*

$$Rf(\bar{\omega}, h) = \int_{S_{\bar{\omega}}^{n-1}} f\left(\omega, \bar{\mu}\left(\frac{\tan h}{\langle \omega, \bar{\omega} \rangle}\right)\right) \frac{(\langle \omega, \bar{\omega} \rangle^2 \cot^2 h + 1)^{-n/2}}{\sin h} d\omega,$$

where $d\omega$ is the surface measure of S^{n-1} , $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product and $S_{\bar{\omega}}^{n-1} = \{\omega \in S^{n-1}: 0 \leq \langle \omega, \bar{\omega} \rangle\}$. (Here and below we use the shorthand $Rf(\bar{\omega}, h) = Rf(\xi(\bar{\omega}, h))$.)

Proof. First we assume that $n = 2$ and parameterize S^1 by an angle, α , with respect to some fixed direction. Thus we can write immediately that

$$Rf(\bar{\omega}, h) = \int_{-\pi/2}^{\pi/2} f(\alpha + \bar{\alpha}, x(\alpha)) \frac{d\alpha}{d\alpha},$$

where $\bar{\alpha}$ is the angle of $\bar{\omega}$ with respect to the fixed direction and $x(\alpha), d(\alpha)$ come from (1.1) and (1.2). It is straightforward from (1.2) that

$$Rf(\bar{\omega}, h) = \int_{-\pi/2}^{\pi/2} f\left(\alpha + \bar{\alpha}, \bar{\mu}\left(\frac{\tan h}{\cos \alpha}\right)\right) \frac{1/\sin h}{1 + \cos^2 \alpha \cot^2 h} d\alpha.$$

For the case $n > 2$ the relevant configuration can be obtained by rotating Figure 2 around the straight line OH . It is well known that the surface element of a geodesic sphere with radius ϱ is $\sin^{n-1} \varrho d\omega$. Since $\xi(\bar{\omega}, h)$ is rotational submanifold the surface measure on $\xi(\bar{\omega}, h)$ at the point X reads as $\sin^{n-2} x(\alpha) \frac{dd}{d\alpha} \Big|_{\cos \alpha = \langle \bar{\omega}, \omega \rangle} d\omega$. This tells us that

$$Rf(\bar{\omega}, h) = \int_{S_{\bar{\omega}}^{n-1}} f\left(\omega, \bar{\mu}\left(\frac{\tan h}{\langle \omega, \bar{\omega} \rangle}\right)\right) \frac{\sin^{n-2} \left(\bar{\mu}\left(\frac{\tan h}{\langle \omega, \bar{\omega} \rangle}\right)\right)}{\sin h (1 + \langle \omega, \bar{\omega} \rangle^2 \cot^2 h)} d\omega.$$

Calculation of the nominator leads to the statement. ■

Lemma 1.2. *Let $f \in L^2(\Sigma^n)$. Then its boomerang transform is*

$$Bf(\bar{\omega}, x) = \int_{S_{\bar{\omega}}^{n-1}} f(\omega, \bar{\mu}(\langle \omega, \bar{\omega} \rangle \tan x)) \frac{(1 + \langle \omega, \bar{\omega} \rangle^2 \tan^2 x)^{-1}}{\cos x} d\omega.$$

Proof. Assuming first that the dimension $n = 2$ one can write immediately that

$$Bf(\bar{\omega}, x) = \int_{-\pi/2}^{\pi/2} f(\alpha + \bar{\alpha}, h(\alpha)) \frac{d\beta}{d\alpha} d\alpha,$$

where $h(\alpha), \beta(\alpha)$ come from (1.1) and (1.3). Then (1.3) gives

$$Bf(\bar{\omega}, x) = \int_{-\pi/2}^{\pi/2} f(\alpha + \bar{\alpha}, \bar{\mu}(\cos \alpha \tan x)) \frac{1/\cos x}{1 + \cos^2 \alpha \tan^2 x} d\alpha.$$

For higher dimension the situation on Figure 2 is simply rotated around the straight line OX so the measures on the unit spheres in the tangent spaces $T_X \Sigma^n$ and $T_O \Sigma^n$ differ only in the coefficient $\frac{d\beta}{d\alpha}$ that proves the lemma. ■

2. Inversion formulas and support theorems

We need the following two technical lemmas that can be easily proven from the formulas given in [2] and [3].

Lemma 2.1. *If $m \in \mathbb{Z}$ then*

$$\frac{\pi}{2} = \int_t^q \frac{\cos(m \arccos(\tan h / \tan q))}{\sqrt{1 - \tan^2 h / \tan^2 q}} \times \frac{\cosh(\operatorname{marccosh}(\tan h / \tan t))}{\sqrt{\tan^2 h / \tan^2 t - 1}} \frac{1/\sin h}{\cos h} dh.$$

Lemma 2.2. *If $m \in \mathbb{Z}$, $n > 2$, $\lambda = (n - 2)/2$ and C_m^λ denotes the Gegenbauer polynomials of the first kind, then*

$$\begin{aligned} M \left(\frac{\sin(q - t)}{\sin q \sin t} \right)^{n-2} &= \int_t^q \cot^{n-3} h C_m^\lambda \left(\frac{\tan h}{\tan t} \right) C_m^\lambda \left(\frac{\tan h}{\tan q} \right) \times \\ &\quad \times \left(\frac{\tan^2 h}{\tan^2 t} - 1 \right)^{\frac{n-3}{2}} \left(1 - \frac{\tan^2 h}{\tan^2 q} \right)^{\frac{n-3}{2}} \frac{dh}{\sin^2 h}, \end{aligned}$$

where

$$M = \pi 2^{3-n} \left(\frac{\Gamma(m + n - 2)}{\Gamma(m + 1)\Gamma(\lambda)} \right)^2 \frac{1}{\Gamma(n - 1)}.$$

Now we present two propositions that describe our transformations in terms of spherical harmonics. The most important facts we need are the following.

A complete orthonormal system in the Hilbert space $L^2(S^{n-1})$ can be chosen consisting of spherical harmonics $Y_{l,m}$, where $Y_{l,m}$ is of degree m . If $Y_{l,m}$ is a member of such a system, $f \in C^\infty(S^{n-1} \times [0, \infty))$ and $p \in [0, \infty)$ let the corresponding coefficients of the series in this system for $f(\omega, p)$ be $f_{l,m}(p)$. Then the series

$$\sum_{l,m} f_{l,m}(p) Y_{l,m}(\omega)$$

converges uniformly absolutely on the compact subsets of $S^{n-1} \times [0, \infty)$ to $f(\omega, p)$ [18].

Below we use the expansions

$$f(\varphi, q) = \sum_{m=-\infty}^{\infty} f_m(q) \exp(im\varphi) \quad \text{and} \quad f(\omega, q) = \sum_{l,m} f_{l,m}(q) Y_{l,m}(\omega)$$

in dimension 2 and in higher dimensions, respectively. These expansions will be used for the Radon and boomerang transforms Rf and Bf as well.

Proposition 2.3.

i) If $f(\varphi, p) \in L^2(\Sigma^2) \cong L^2(S^1 \times [0, \frac{\pi}{2}))$ then

$$(r) \quad (Rf)_m(h) = 2 \int_h^{\pi/2} f_m(q) \frac{\cos(m \arccos(\tan h / \tan q))}{\cos h \sqrt{1 - \tan^2 h / \tan^2 q}} dq.$$

ii) If $n > 2$, $f(\omega, p) \in L^2(\Sigma^n) \cong L^2(S^{n-1} \times [0, \frac{\pi}{2}))$ and $\lambda = (n-2)/2$ then

$$(rn) \quad (Rf)_{l,m}(h) = \frac{|S^{n-2}|}{C_m^\lambda(1)} \int_h^{\pi/2} f_{l,m}(q) C_m^\lambda\left(\frac{\tan h}{\tan q}\right) \left(1 - \frac{\tan^2 h}{\tan^2 q}\right)^{\frac{n-3}{2}} \frac{\sin^{n-2} q}{\cos h} dq,$$

where $|S^k|$ is the area of S^k .

Proof. If $n = 2$ substituting the expansions of f and Rf into the formula of Lemma 1.1. we get

$$(Rf)_m(h) = \int_{-\pi/2}^{\pi/2} f_m\left(\bar{\mu}\left(\frac{\tan h}{\cos \alpha}\right)\right) \frac{1/\sin h}{1 + \cos^2 \alpha \cot^2 h} \exp(im\alpha) d\alpha.$$

Putting $q = \bar{\mu}(\frac{\tan h}{\cos \alpha})$ ($\cos \alpha = \tan h / \tan q$) this becomes (r).

Writing the expansions of f and Rf into the formula of Lemma 1.1 the Funk-Hecke theorem [21] gives

$$(Rf)_{l,m}(h) = \frac{|S^{n-2}|}{C_m^\lambda(1)} \int_0^1 f_{l,m}\left(\bar{\mu}\left(\frac{\tan h}{t}\right)\right) C_m^\lambda(t) (1-t^2)^{\frac{n-3}{2}} \frac{(t^2 \cot^2 h + 1)^{-n/2}}{\sin h} dt.$$

Putting $q = \bar{\mu}(\frac{\tan h}{t})$ ($t = \tan h / \tan q$) this becomes (rn) which was to be proved. ■

Proposition 2.4.

i) If $f(\varphi, p) \in L^2(\Sigma^2) \cong L^2(S^1 \times [0, \frac{\pi}{2}))$ then

$$(b) \quad (Bf)_m(h) = 2 \int_0^h f_m(q) \frac{\cos(m \arccos(\tan q / \tan h))}{\sin h \sqrt{1 - \tan^2 q / \tan^2 h}} dq.$$

ii) If $n > 2$, $f(\omega, p) \in L^2(\Sigma^n) \cong L^2(S^{n-1} \times [0, \frac{\pi}{2}))$ and $\lambda = (n-2)/2$ then

$$(bn) \quad (Bf)_{l,m}(h) = \frac{|S^{n-2}|}{C_m^\lambda(1)} \int_0^h f_{l,m}(q) C_m^\lambda\left(\frac{\tan q}{\tan h}\right) \left(1 - \frac{\tan^2 q}{\tan^2 h}\right)^{\frac{n-3}{2}} \frac{dq}{\sin h}.$$

This proposition can be proved from Lemma 1.2 in the same way as the previous one so we leave the proof to the reader.

Our following theorems give the inversion formulas. Since their proofs are very similar we give only the first one's proof.

Theorem 2.5.

i) If $f \in C^\infty(\Sigma^2) \cong C^\infty(S^1 \times [0, \frac{\pi}{2}))$ then

$$(ri) \quad f_m(t) = \frac{1}{\pi} \frac{d}{dt} \int_t^{\pi/2} (Rf)_m(h) \frac{\cosh(m \operatorname{arccosh}(\tan h / \tan t))}{\sin h \sqrt{\tan^2 h / \tan^2 t - 1}} dh.$$

ii) If $n > 2$, $f \in C^\infty(\Sigma^n) \cong C^\infty(S^{n-1} \times [0, \frac{\pi}{2}))$ then

$$(rni) \quad f_{l,m}(t) = (-1)^{n-1} \frac{\Gamma(m+1)\Gamma(\lambda)}{2\pi^{n/2}\Gamma(m+n-2)} \begin{cases} \frac{d}{dt} \delta_2 \delta_4 \cdots \delta_{n-2} F(t) & \text{if } n \text{ even} \\ \delta_1 \delta_3 \delta_5 \cdots \delta_{n-2} F(t) & \text{if } n \text{ odd} \end{cases},$$

where

$$(rni') \quad F(t) = \int_t^{\pi/2} (Rf)_{l,m}(h) C_m^\lambda\left(\frac{\tan h}{\tan t}\right) \left(\frac{\tan^2 h}{\tan^2 t} - 1\right)^{\frac{n-3}{2}} \frac{\sin^{n-2} t}{\sin h} \cot^{n-2} h dh$$

and $\delta_k = \frac{d^2}{dt^2} + k^2$ ($k \in \mathbb{N}$).

Theorem 2.6.

i) If $f \in C^\infty(\Sigma^2) \cong C^\infty(S^1 \times [0, \frac{\pi}{2}))$ then

$$(bi) \quad f_m(t) = \frac{1}{\pi} \frac{d}{dt} \int_0^t (Bf)_m(h) \frac{\cosh(\operatorname{marccosh}(\tan t / \tan h))}{\cos h \sqrt{\tan^2 t / \tan^2 h - 1}} dh.$$

ii) If $n > 2$, $f \in C^\infty(\Sigma^n) \cong C^\infty(S^{n-1} \times [0, \frac{\pi}{2}))$ then

$$(bni) \quad f_{l,m}(t) = (-1)^{n-1} \frac{\Gamma(m+1)\Gamma(\lambda)}{2\pi^{n/2}\Gamma(m+n-2)} \cos^{n-2} t \times \\ \times \left\{ \begin{array}{ll} \frac{d}{dt} \delta_2 \delta_4 \cdots \delta_{n-2} F(t) & \text{if } n \text{ even} \\ \delta_1 \delta_3 \delta_5 \cdots \delta_{n-2} F(t) & \text{if } n \text{ odd} \end{array} \right\},$$

where

$$(bni') \quad F(t) = \int_0^t (Bf)_{l,m}(h) C_m^\lambda \left(\frac{\tan t}{\tan h} \right) \left(\frac{\tan^2 t}{\tan^2 h} - 1 \right)^{\frac{n-3}{2}} \frac{\cos^{n-2} t}{\cos h} \tan^{n-2} h dh.$$

Proof. To prove (ri) one multiplies (r) by

$$\frac{\cosh(\operatorname{marccosh}(\tan h / \tan t))}{\sin h \sqrt{\tan^2 h / \tan^2 t - 1}}$$

and integrates from t to $\pi/2$. Then one simplifies the integral using Lemma 2.1 and finally differentiates with respect to t .

To prove (rni) one multiplies (rn) by

$$C_m^\lambda \left(\frac{\tan h}{\tan t} \right) \left(\frac{\tan^2 h}{\tan^2 t} - 1 \right)^{\frac{n-3}{2}} \frac{\sin^{n-2} t}{\sin h} \cot^{n-2} h$$

and integrates from t to $\pi/2$ again. Then Lemma 2.2 leads to

$$F(t) = M \frac{|S^{n-2}|}{C_m^\lambda(1)} \int_t^{\pi/2} f_{l,m}(q) \sin^{n-2}(q-t) dq.$$

To finish the proof it is enough to observe that

$$\frac{d^2}{dt^2} \sin^k(q-t) = -k^2 \sin^k(q-t) + k(k-1) \sin^{k-2}(q-t). \quad \blacksquare$$

The following two corollaries are direct consequences of the above theorems.

Corollary 2.7. *If $f \in C_c^\infty(\Sigma^n)$ and $\pi/2 > A > 0$ then the values of $Rf(\omega, p)$ for $\pi/2 > p \geq A$ determine $f(\omega, p)$ for $\pi/2 > p \geq A$. If $Rf(\omega, p) = 0$ on this domain, then $f(\omega, p) = 0$ too.*

Corollary 2.8. *If $f \in C^\infty(\Sigma^n)$ and $\pi/2 > A > 0$ then the values of $Bf(\omega, p)$ for $0 \leq p \leq A$ determine $f(\omega, p)$ for $0 \leq p \leq A$. If $Bf(\omega, p) = 0$ on this domain, then $f(\omega, p) = 0$ too.*

3. Null spaces and ranges

Our first proposition in this section establishes the continuity of the Radon and boomerang transforms. Let $\psi(\omega, p)$ denote the surface from the points of which the geodesic segment joining O and $\bar{\mu} \circ i_\Sigma(\omega, p)$ can be seen at 90° degrees.

Proposition 3.1. *Let S be a measurable set in Σ^n , $n \geq 3$. The maps*

$$R: L^2(S, \sin^{n-1} \delta_x dx) \rightarrow L^2(S^R) \quad \text{and} \quad B: L^2(S, \sin^{1-n} \delta_x dx) \rightarrow L^2(S^B)$$

are continuous, where

$$S^R = \{(\omega, p) \in S^{n-1} \times [0, \pi/2): \xi(\omega, p) \cap S \neq \emptyset\}$$

$$S^B = \{(\omega, p) \in S^{n-1} \times [0, \pi/2): \psi(\omega, p) \cap S \neq \emptyset\},$$

δ_x is the distance of x from the origin and dx is the Lebesgue measure on Σ^n .

Proof. We can assume $S = \Sigma^n$ without loss of generality. By the orthogonality of the spherical harmonics $Y_{l,m}$ it is enough to prove for $f(\omega, p) = h(p)Y_{l,m}(\omega)$ that

$$\|Rf\|_{L^2(S^{n-1} \times [0, \pi/2))}^2 \leq 16|S^{n-2}|^2 \|f\|_{L^2(\Sigma^n, \sin^{n-1} \delta_x dx)}^2.$$

Using (rn) we see that

$$\begin{aligned} \|Rf\|_{L^2(S^{n-1} \times [0, \pi/2))}^2 &= 4\|(Rf)_{l,m}\|_{L^2([0, \pi/2))}^2 = \frac{4|S^{n-2}|^2}{(C_m^\lambda(1))^2} \times \\ &\times \int_0^{\pi/2} \frac{1}{\cos^2 h} \left(\int_h^{\pi/2} h(q) C_m^\lambda \left(\frac{\tan h}{\tan q} \right) \left(1 - \frac{\tan^2 h}{\tan^2 q} \right)^{\frac{n-3}{2}} \sin^{n-2} q dq \right)^2 dh. \end{aligned}$$

Since $|C_m^\lambda(t)| \leq |C_m^\lambda(1)|$ for $t \in [0, 1]$ and $|1 - \tan^2 h / \tan^2 q| \leq 1$ we get

$$\begin{aligned} & \|Rf\|_{L^2(S^{n-1} \times [0, \pi/2))}^2 \\ & \leq 4|S^{n-2}|^2 \int_0^{\pi/2} \frac{1}{\cos^2 h} \left(\int_h^{\pi/2} |h(q)| \sin^{n-2} q \, dq \right)^2 dh \\ & = 4|S^{n-2}|^2 \int_0^\infty \left(\frac{1}{x} \int_0^x |h(\arctan(\frac{1}{y}))| \sin^{n-2}(\arctan(\frac{1}{y})) \frac{dy}{1+y^2} \right)^2 dh \end{aligned}$$

with the substitutions $x = \cot h$ and $y = \cot q$. At the same time Hardy's inequality

$$\left\| \frac{1}{v} \int_0^v g(u) \, du \right\|_{L^2(\mathbb{R}_+)} \leq 2\|g\|_{L^2(\mathbb{R}_+)}$$

gives

$$\begin{aligned} & \|Rf\|_{L^2(S^{n-1} \times [0, \pi/2))}^2 \\ & \leq 16|S^{n-2}|^2 \int_0^\infty \left(\left| h\left(\arctan\left(\frac{1}{y}\right)\right) \right| \frac{\sin^{n-2}\left(\arctan\left(\frac{1}{y}\right)\right)}{1+y^2} \right)^2 dy. \end{aligned}$$

Putting $y = \cot z$ we find the theorem for R . The proof for the boomerang transform is very similar and it is left to the reader. ■

Theorem 3.2. *Let S be a compact set in Σ^n and $n \geq 3$. The Radon transform is an injection of $L^2(S, \sin^{n-1} \delta_x dx)$ into*

$$\mathcal{A} = \left(\text{Cl Sp} \left\{ g_{j,l,m}(\omega, h) = \frac{\tan^j h}{\cos h} Y_{l,m}(\omega) : 0 \leq j < m \text{ } (m-j) \text{ is odd} \right\} \right)^\perp \cap L^2(S^R),$$

where Cl Sp means the closure of the span of the set of functions indicated.

Proof. $g_{j,l,m} \in L_c^2(S^{n-1} \times [0, \pi/2))$ is proved by a simple integration using the fact that it is of compact support. Let $f(\omega, h) = v(h)Y_{i,k}(\omega) \in L^2(S, \sin^{n-1} \delta_x dx)$ and $g^*(\omega, h) = g(\tan h)Y_{l,m}(\omega)/\cos h$. We get immediately from (rn) by changing the order of integrations that

$$\begin{aligned} \langle g^*, Rf \rangle &= \delta_{l,i} \delta_{m,k} \frac{4|S^{n-2}|}{C_k^\lambda(1)} \int_0^{\pi/2} v(q) \frac{\sin^{n-1} q}{\cos q} \times \\ & \quad \times \int_0^1 g(x \tan q) C_k^\lambda(x) \left(1 - x^2\right)^{\frac{n-3}{2}} dx \, dq, \end{aligned}$$

where $\delta_{m,k}$ is the Cronecker delta and $\langle \cdot, \cdot \rangle$ is the inner product in $L_c^2(S^{n-1} \times [0, \pi/2))$. This means that Rf is orthogonal to g^* for all $f \in L^2(S, \sin^{n-1} \delta_x dx)$ if and only if

$$0 \equiv \int_0^1 g(xy) C_k^\lambda(x) (1-x^2)^{\frac{n-3}{2}} dx.$$

By the Lemma 5.1 and Theorem 5.2 of [16] and Theorem 3.2 of [17] this is equivalent to g being in the closure of the span of functions x^i , where $0 \leq i < k$ and $k-i$ is even. This proves that the range of the Radon transform is in \mathcal{A} .

Now we prove the injectivity of R . We are looking for a function $f(\omega, h) = v(h)Y_{i,k}(\omega) \in L^2(S, \sin^{n-1} \delta_x dx)$, which has zero Radon transform. Thus we should find a function $v \in L^2([0, \pi/2), \sin^{2n-2} q dq)$ which satisfies the equation

$$0 \equiv \int_h^{\pi/2} v(q) C_m^\lambda\left(\frac{\tan h}{\tan q}\right) \left(1 - \frac{\tan^2 h}{\tan^2 q}\right)^{\frac{n-3}{2}} \sin^{n-2} q dq.$$

Assuming $v(q) = g(\cot q) \sin^{1-n} q / \cos q$ and changing the variables we get the integral equation

$$0 \equiv \int_t^\infty \frac{1}{s} g\left(\frac{1}{s}\right) C_m^\lambda\left(\frac{t}{s}\right) \left(1 - \frac{t^2}{s^2}\right)^{\frac{n-3}{2}} ds,$$

which has to be satisfied by g for $t \in [0, \infty)$. Since $g(\frac{1}{s}) \in L_c^2([0, \infty))$ and S is compact this Volterra integral equation is of the type (3.7) of [19], so its only solution is the zero function, which completes the proof. ■

Similar methods can be used for getting the corresponding results for the boomerang transform. We note only the most interesting one.

Theorem 3.3. *Let S be a compact set in Σ^n and $n \geq 3$. The boomerang transform has kernel in $L^2(S, \sin^{1-n} \delta_x dx)$*

$$\text{Cl Sp} \left\{ g_{j,l,m}(\omega, h) = \frac{\tan^j h}{\cos^2 h} Y_{l,m}(\omega) : 0 \leq j < m \text{ \& } (m-j) \text{ is odd} \right\} \subset L^2(S^B).$$

4. Closed inversion formulas

Theorem 4.1. *For $n \geq 2$ and $f \in C^\infty(\Sigma^n)$ if n is odd then*

$$f(\bar{\omega}, t) = (-1)^{\frac{n-1}{2}} \frac{2^{1-n}}{\pi^{n-1}} \delta_1 \delta_3 \cdots \delta_{n-2} \left(B \left(Rf(\omega, h) \frac{\cot^{2\lambda} h}{\sin h} \right) (\bar{\omega}, t) \sin^{n-1} t \right).$$

If n is even then

$$f(\bar{\omega}, t) = (-1)^{\frac{n-2}{2}} \frac{2^{1-n}}{\pi^n} \frac{d}{dt} \delta_2 \delta_4 \cdots \delta_{n-2} \left(B \left(\mathcal{H} \left\langle Rf(\omega, h) \frac{\cot^{2\lambda} h}{\sin h} \right\rangle \right) (\bar{\omega}, t) \sin^{n-1} t \right),$$

where the \mathcal{H} distribution is

$$\mathcal{H}\langle f \rangle(\omega, h) = \frac{1}{\cos^2 h} \int_{-\pi/2}^{\pi/2} f(\omega, r) \frac{1}{\tan r - \tan h} dr.$$

Proof. We start with the case of odd dimension, when Theorem 2.5 tells us that

$$f_{l,m}(t) = \mathcal{CD} \int_t^{\pi/2} (Rf)_{l,m}(h) C_m^\lambda \left(\frac{\tan h}{\tan t} \right) \left(\frac{\tan^2 h}{\tan^2 t} - 1 \right)^{\frac{n-3}{2}} \frac{\sin^{n-2} t}{\sin h} \cot^{n-2} h dh,$$

where $\mathcal{C} = \frac{\Gamma(m+1)\Gamma(\lambda)}{2\pi^{n/2}\Gamma(m+n-2)}$ and $\mathcal{D} = \delta_1 \delta_3 \cdots \delta_{n-2}$. The integral $\int_t^{\pi/2}$ can be modified by making use of $\int_t^{\pi/2} = \int_0^{\pi/2} - \int_0^t$ that yields

$$(*) \quad f_{l,m}(t) = I + (-1)^{\frac{n-1}{2}} \mathcal{CD} \int_0^t (Rf)_{l,m}(h) C_m^\lambda \left(\frac{\tan h}{\tan t} \right) \times \\ \times \left(1 - \frac{\tan^2 h}{\tan^2 t} \right)^{\frac{n-3}{2}} \frac{\sin^{n-2} t}{\sin h} \cot^{n-2} h dh,$$

where

$$I = \mathcal{CD} \int_0^{\pi/2} (Rf)_{l,m}(h) C_m^\lambda \left(\frac{\tan h}{\tan t} \right) \left(\frac{\tan^2 h}{\tan^2 t} - 1 \right)^{\frac{n-3}{2}} \frac{\sin^{n-2} t}{\sin h} \cot^{n-2} h dh.$$

Using (rn) and reversing the order of integrations one gets that I is proportional to

$$\int_0^{\pi/2} f_{l,m}(q) \sin^{2\lambda} q \int_0^q C_m^\lambda \left(\frac{\tan h}{\tan q} \right) \left(1 - \frac{\tan^2 h}{\tan^2 q} \right)^{\frac{n-3}{2}} \times \\ \times \mathcal{D} \left(C_m^\lambda \left(\frac{\tan h}{\tan t} \right) \left(\frac{\tan^2 h}{\tan^2 t} - 1 \right)^{\frac{n-3}{2}} \sin^{2\lambda} t \right) \frac{\cot^{n-1} h}{\cos^2 h} dh dq.$$

Substitution of the variable $x = \tan h / \tan q$ in the inner integral J and using that the integrand is an even function we obtain

$$J = \frac{\cot^{2\lambda} q}{2} \int_{-1}^1 C_m^\lambda(x) (1-x^2)^{\frac{n-3}{2}} \times \\ \times \mathcal{D} \left(C_m^\lambda \left(\frac{\tan q}{\tan t} x \right) \left(\frac{\tan^2 q}{\tan^2 t} x^2 - 1 \right)^{\frac{n-3}{2}} \sin^{2\lambda} t \right) x^{1-n} dx.$$

Let us recall that $C_m^\lambda(x)(1-x^2)^{\frac{n-3}{2}}$ is a polynomial of degree $m+n-3$ and $\{C_m^\lambda(x)\}$ is orthogonal polynomial system on $[-1, 1]$ with weight-function $(1-x^2)^{\frac{n-3}{2}}$ [8]. Now we prove that

$$\mathcal{D} \left(C_m^\lambda \left(x \frac{\tan q}{\tan t} \right) \left(x^2 \frac{\tan^2 q}{\tan^2 t} - 1 \right)^{\frac{n-3}{2}} \sin^{2\lambda} t \right)$$

is homogeneous of degree $n-1$ in x , which implies $J = 0$ and $I = 0$ by the above facts. The coefficient of x^k , for $0 \leq k \leq n-2$, vanishes if

$$\delta_1 \delta_3 \cdots \delta_{n-2} \left(\sin^{n-2} t \cot^k t \right) = 0,$$

that can be verified by simple induction after establishing that

$$\delta_k(\cos^k t) = -k(k-1) \cos^{k-2} t \quad \text{and} \quad \delta_k(\sin^k t) = k(k-1) \sin^{k-2} t.$$

Now the equation (*) gives just the expansion of the closed inversion formula stated for odd dimension.

Suppose the dimension n is now even and λ is integer. Let \mathcal{D} denote the differential operator $\frac{d}{dt} \delta_2 \delta_4 \cdots \delta_{n-2}$. Theorem 2.5 says
(**)

$$f_{l,m}(t) = -\mathcal{CD} \int_t^{\pi/2} (\mathbf{R}f)_{l,m}(h) C_m^\lambda \left(\frac{\tan h}{\tan t} \right) \left(\frac{\tan^2 h}{\tan^2 t} - 1 \right)^{\frac{n-3}{2}} \frac{\sin^{n-2} t}{\sin h} \cot^{n-2} h dh.$$

From now on we will frequently use the notions and the numbers of equations of [4], [5] to avoid the long explanations.

Analogously to the odd-dimensional case, one can easily see that

$$0 = \mathcal{CD} \int_0^{\pi/2} (\mathbf{R}f)_{l,m}(h) E_{m+2\lambda-1}^\lambda \left(\frac{\tan h}{\tan t} \right) \frac{\sin^{n-2} t}{\sin h} \cot^{n-2} h dh,$$

since $E_{m+2\lambda-1}^\lambda$ is a polynomial of degree $m+n-3$ [4]. After separating this integral as $\int_0^t + \int_t^{\pi/2}$, substituting (A.14) and (A.4) of [4] into \int_0^t and $\int_t^{\pi/2}$ respectively one

can proceed by adding the result to the equation (**), which results in

$$\begin{aligned} f_{l,m}(t) &= -\mathcal{CD} \int_t^{\pi/2} (\mathbf{R}f)_{l,m}(h) 2D_m^\lambda \left(\frac{\tan h}{\tan t} \right) \left(\frac{\tan^2 h}{\tan^2 t} - 1 \right)^{\frac{n-3}{2}} \frac{\sin^{n-2} t}{\sin h} \cot^{n-2} h \, dh - \\ &\quad - \mathcal{CD} \int_0^t (\mathbf{R}f)_{l,m}(h) (-1)^\lambda D_m^\lambda \left(\frac{\tan h}{\tan t} \right) \left(1 - \frac{\tan^2 h}{\tan^2 t} \right)^{\frac{n-3}{2}} \frac{\sin^{n-2} t}{\sin h} \cot^{n-2} h \, dh. \end{aligned}$$

Using (24) and (25) of [4] this leads to

$$f_{l,m}(t) = -\frac{(-1)^\lambda}{\pi} \mathcal{CD} \int_0^{\pi/2} (\mathbf{R}f)_{l,m}(h) I_m^\lambda \left(\frac{\tan h}{\tan t} \right) \frac{\sin^{n-2} t}{\sin h} \cot^{n-2} h \, dh,$$

where by (22) of [4]

$$I_m^\lambda(y) = \int_{-1}^1 C_m^\lambda(x) (1-x^2)^{\frac{n-3}{2}} (y-x)^{-1} \, dx.$$

Now the substitution $x = \tan r / \tan t$ and a change in the order of integrations give

$$\begin{aligned} f_{l,m}(t) &= \frac{-(-1)^\lambda}{\pi} \mathcal{CD} \int_{-t}^t C_m^\lambda \left(\frac{\tan r}{\tan t} \right) \left(1 - \frac{\tan^2 r}{\tan^2 t} \right)^{\frac{n-3}{2}} \frac{\sin^{n-1} t}{\sin t \cos^2 r} \times \\ &\quad \times \int_0^{\pi/2} (\mathbf{R}f)_{l,m}(h) \frac{\cot^{n-2} h}{\tan h - \tan r} \frac{dh}{\sin h} \, dr. \end{aligned}$$

This equation is just the expansion of the closed inversion formula stated for even dimension. ■

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